

ON THE LARGE AMPLITUDE MOTION OF  
TUNED PENDULUM DAMPERS

A THESIS

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by

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TUNED PENDULUM DAMPERS

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## LIST OF SYMBOLS

- a - Distance from rotor shaft axis to pendulum pivot axis
- b - Number of blades in rotor
- $C_1$  - Viscous torque parameter
- $m_1$  - Mass of a pendulum damper
- m - Mass of equivalent pendulum damper
- n - Number of pendulums
- $\bar{r}$  - Position vector
- t - Time
- $t'$  - Non-dimensional time
- u - Dummy Variable
- v - Dummy Variable
- A - Non-dimensional pendulum inertia parameter
- B - Non-dimensional pendulum mass parameter
- C - Non-dimensional forcing function
- D - Non-dimensional pivot radius
- $D_t(\Theta_t)$  - Generalized dissipative torque
- E - Non-dimensional damping factor
- $\bar{F}$  - Force applied to a particle
- $I(\epsilon)$  - Total change in energy as a function of  $\epsilon$
- $I_A$  - Mass moment of inertia of pendulum about pivot axis
- $K_i$  - Constant of integration
- $N(t)$  - Generalized forcing function



- $R_A$  - Distance from pivot axis to center of gravity of pendulum damper  
 $T$  - Kinetic energy  
 $W(\epsilon_1, \epsilon_2)$  - Work done by force system on equivalent pendulum  
 $\delta$  - Axial displacement of rotor shaft  
 $\Theta$  - Angular displacement of pendulum damper  
 $\Omega$  - Rotor angular velocity  
 $\epsilon_i$  - Variational parameter  
 $\eta_i$  - Variational function  
 $\bar{m}$  - Mass of helicopter including pendulum mass  
 $\Theta_t$  - Oscillatory velocity of pendulum dampers  
 $\Theta_{tt}$  - Angular acceleration of pendulum dampers  
 $\delta_t$  - Axial disturbance velocity of rotor shaft  
 $\delta_{tt}$  - Axial acceleration of rotor shaft  
 $\mathcal{P}$  - Position radius vector  
 $\tau$  - Computer time

## SUMMARY

General equations of motion were developed for pendulum dampers of arbitrary geometry mounted on a rotating shaft. Solutions to these equations were subsequently obtained on an analog computer.

The equations of motion were derived for a non-conservative force system in which the pendulum dampers were attached to a pivot arm mounted on a rotating shaft and were allowed to oscillate through large amplitude displacements. The non-linear equations of motion for the system were then non-dimensionalized and arranged in a form readily applicable to computer programming. Because the parameters appeared in non-dimensional form in the equations of motion, an infinite variety of tuned pendulum damper configurations can be obtained from any combination of parameters that minimize the axial vibration of the rotating shaft.

In particular, two non-dimensional solutions were obtained for light-weight pendulum dampers tuned to remove a twice per revolution axial excitation in the rotor drive shaft of a helicopter. The large amplitude tuning with damping was found to be quite sharp. The corresponding viscous torque required in the pivot axes of the large amplitude pendulum dampers was relatively small, and it is believed that the heat dissipation problem would not be serious.

Almost a one hundred percent reduction in the axial vibration was obtained for the tuned pendulum dampers having a peak oscillatory amplitude of  $\pm 1.1$  radians. Approximately sixty-five percent of the axial shaft vibration was removed for the pendulum dampers with peak oscillatory amplitudes of  $\pm \pi/6$  radians.

## CHAPTER I

### INTRODUCTION

Periodic excitations imparted axially to a rotating shaft occur in many dynamic systems, such as rotor drive shafts of helicopters, propeller drive shafts, the axles of automobile wheels, and other similar rotating mechanisms. These axial excitations are commonly called longitudinal excitations.

Longitudinal excitations in a rotating shaft can cause serious structural problems, for example, the vertical excitations imparted through the rotor drive shaft to the air frame of a helicopter. This example is of particular interest to aeronautical engineers in the design of high speed helicopters.

At high forward speeds excessive vertical vibrations occur in a helicopter, usually at a frequency equal to the product of the rotor angular velocity times the number of blades in the rotor. There are several possible ways in which these periodic excitations could be reduced. However, most of the aerodynamic and structural methods of vibratory reduction involve undesirable compromises. One feasible approach to the problem might be to mount light-weight, tuned pendulum dampers on the rotating drive shaft. The dampers would oscillate in a plane passing through the shaft axis and would transmit a periodic force to the shaft in opposition to the exciting force.

The practicality of tuned pendulum dampers for vibration absorption in rotary wing aircraft was investigated by R. B. Gray in 1947<sup>1</sup>. His analysis showed that tuned pendulum dampers having small amplitude oscillations were too heavy for helicopter use. However, by allowing large amplitude oscillations, the corresponding lighter weight pendulum dampers might prove practical for helicopter use.

Large amplitude oscillatory motion is, of course, non-linear, and very little information can be found in the literature on this subject. A fair amount of work has been done on "Spool-type" torque absorbers for internal combustion engines (2) having large amplitude motion, but not on thrust or axial-type absorbers. Some non-linear pendulum problems have been solved by transforming the oscillatory motion from circular to elliptic motion(3).

On the other hand, a pendulum damper of arbitrary shape can be investigated in non-linear circular motion with the aid of present day computers; and, from computer solutions, tuning and amplitude-frequency curves can be obtained. To this end, then, the following analysis is based on a pendulum damper of arbitrary weight, size, and shape. Solutions to the resulting equations of motion appear in non-dimensional form and are valid for all second harmonic tuned pendulum dampers designed to remove longitudinal excitations imparted to rotating shafts.

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<sup>1</sup>Numbers in parenthesis indicate references found in the bibliography.

## CHAPTER II

### DEVELOPMENT OF THE EQUATIONS OF MOTION FOR TUNED PENDULUM DAMPERS

Analysis. Consider a helicopter flying at a high forward speed in level unaccelerated flight. It is desired to eliminate the vertical excitations imparted to the air frame that correspond in harmonic frequency order to the number of blades in the rotor. For a helicopter with  $b$  number of blades in the rotor, the  $b^{\text{th}}$  harmonic thrust variation is normally the largest and the most undesirable from a structural point of view.

Figure 1 shows two pendulum dampers of arbitrary geometry attached to a pivot arm which is rigidly mounted on the rotor drive shaft. At least two pendulum dampers would be required, attached to and rotating with the shaft. The dampers would have to be symmetrically positioned around the drive shaft so as to eliminate any transverse vibrations they might otherwise induce. If the forcing function is of appreciable magnitude, and the mass of the pendulums small, then the pendulum dampers will have to undergo large accelerations to cancel the longitudinal vibrations. The forcing function is the periodic thrust

variation that is transmitted through the rotor hub to the drive shaft, as shown in Figure 2. For light weight pendulum dampers, with pivot axes located at a fixed radius from the rotor drive shaft axis, the desired pendulum inertia force can best be realized if the pendulums oscillate through large amplitude displacements. The pendulums considered here are symmetric about the plane of oscillation.

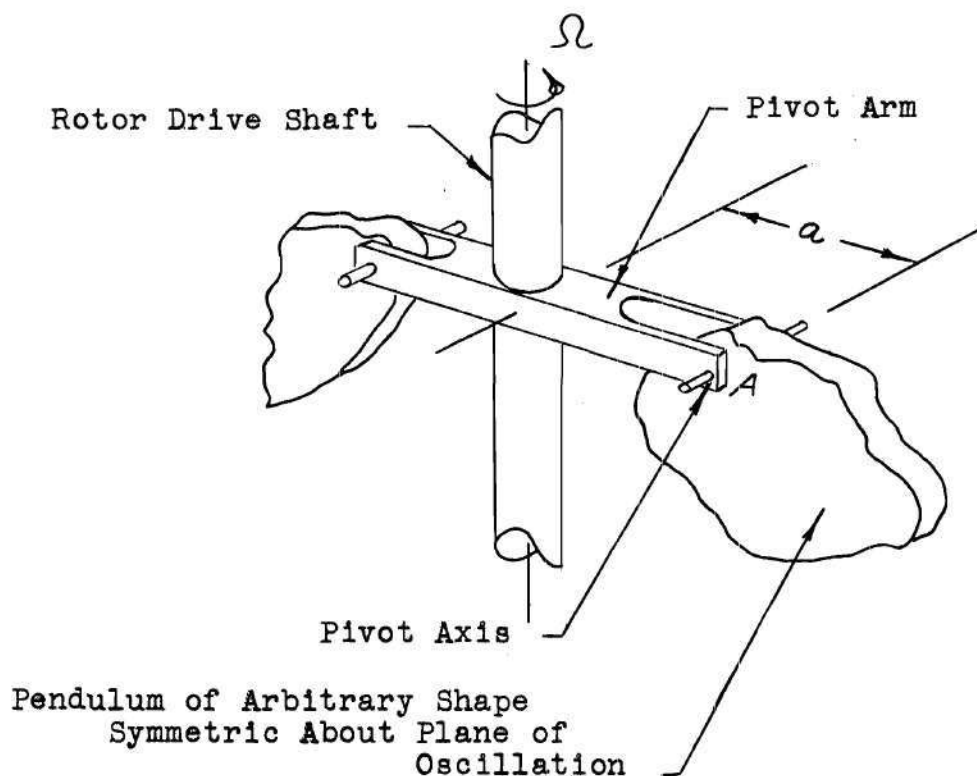


Figure 1. Pendulum Dampers of Arbitrary Geometry Mounted on the Rotor Drive Shaft of a Helicopter.

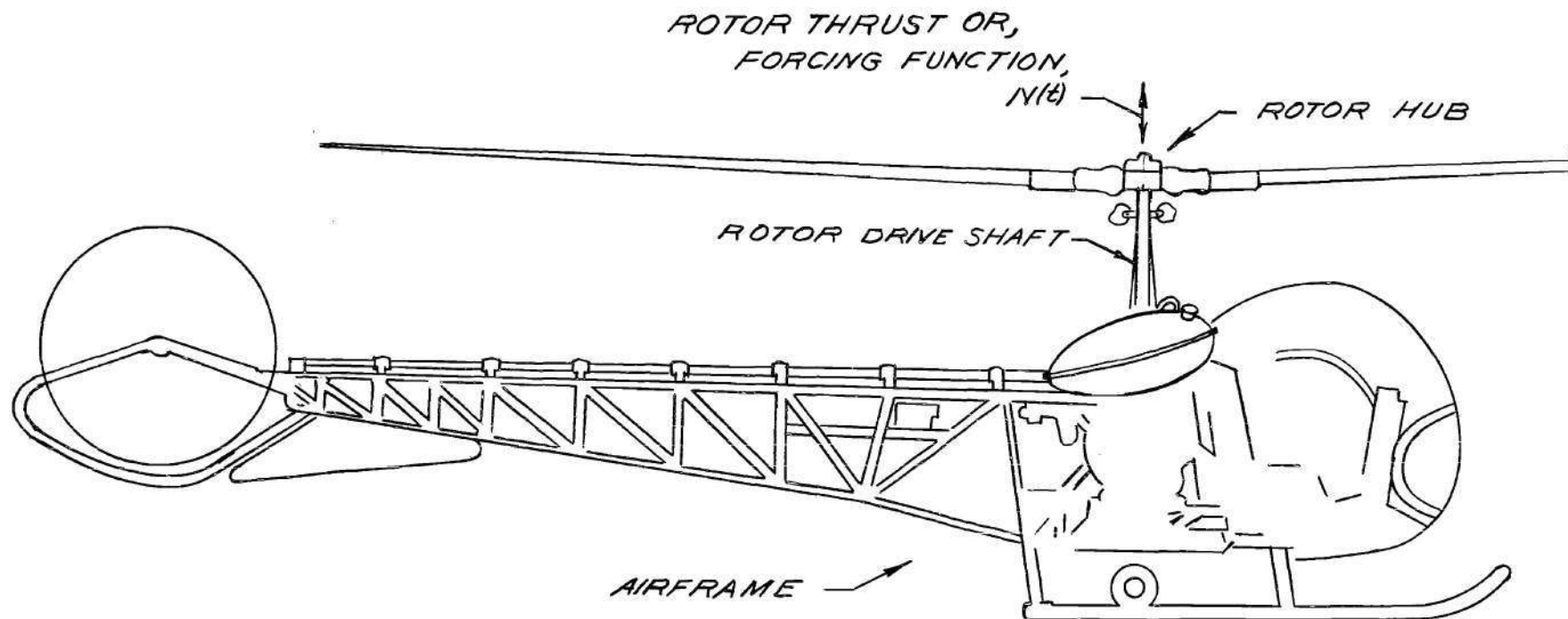


Figure 2. Sketch of Helicopter with Forcing Function Applied Through Rotor Hub to Rotor Drive Shaft.



The following assumptions were made in the development of the equations of motion for the system. Since the inertia of the pendulums is very small compared to the inertia of the rotor drive shaft and the air frame, the Coriolis feed-back torque that would impart a torsional variation in the rotor angular velocity was neglected, and the rotor angular velocity taken as the average angular velocity  $\Omega$ . All structural deflections were neglected and the transmissibility<sup>2</sup> between the shaft and the air frame was taken as unity. The air frame was assumed to be a lumped mass with the rotor drive shaft axis passing through the center of gravity.

The desired form of Hamilton's Integral for a non-conservative force system was obtained as follows<sup>3</sup>. Consider first a single particle of constant mass  $m$ , moving in a non-conservative force field. Define a position vector from a fixed origin to the particle at any time  $t$  as  $\vec{r}(t)$ . Then, according to Newton's second law of motion, the actual path

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<sup>2</sup>Transmissibility is defined as the ratio of the force transmitted from one member, through a connection to a second member, to the force applied to the first member.

<sup>3</sup>A similar development of the Hamilton Integral can be found in reference 4, Chapter 2.8.

followed is governed by the vector equation

$$m \frac{d^2 \bar{r}(t)}{dt^2} - \bar{F} = 0 \quad (1)$$

where  $\bar{F}$  is the force acting on the particle  $m$ . Next, consider any other path  $\bar{r}(t) + \bar{\eta}(t)$ , where  $\bar{\eta}(t)$  is the variation of  $\bar{r}(t)$  as shown in Figure 3. From physical considerations, the true path and the "varied" path must coincide at two distinct times,  $t_1$  and  $t_2$ , that is

$$\bar{\eta}(t_1) = \bar{\eta}(t_2) = 0 \quad (2)$$

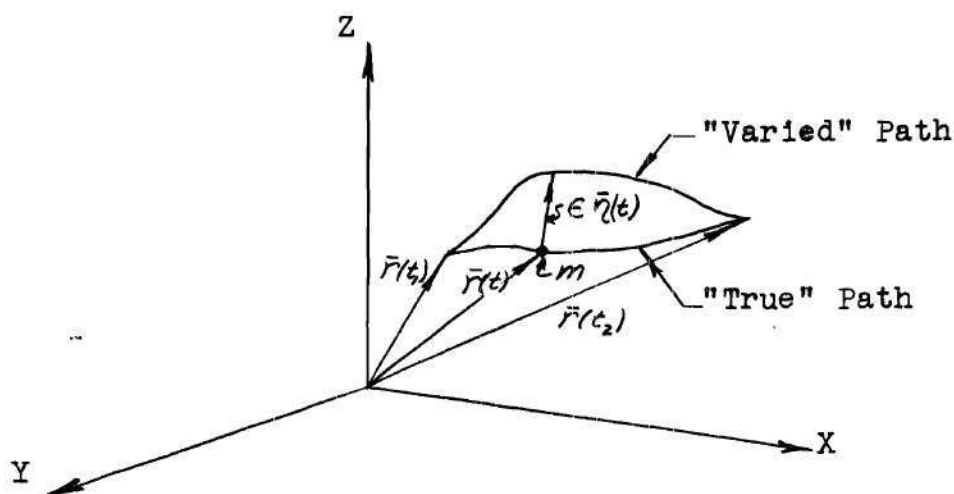


Figure 3. Vector Diagram of Force Acting on a Particle in a Non-conservative Force Field.

The first step in the derivation is to take the scalar, or dot, product of  $\bar{\eta}(t)$  into equation (1) and integrate the result with respect to time from  $t_1$  to  $t_2$  to obtain

$$\int_{t_1}^{t_2} \bar{\eta}(t) \cdot \left\{ m \frac{d^2 \bar{r}}{dt^2} - \bar{F} \right\} dt = 0 \quad (3)$$

Integrating the first term of equation (2) by parts gives

$$m \int_{t_1}^{t_2} \left( \bar{\eta}(t) \cdot \frac{d^2 \bar{r}}{dt^2} \right) dt = m \left\{ \bar{\eta}(t) \cdot \frac{d \bar{r}}{dt} \right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \left( \frac{d \bar{\eta}(t)}{dt} \cdot \frac{d \bar{r}}{dt} \right) dt \quad (4)$$

But by equation (2),  $\bar{\eta}(t_2) = \bar{\eta}(t_1) = 0$ , and equation (4) reduces to

$$m \int_{t_1}^{t_2} \left( \bar{\eta}(t) \cdot \frac{d^2 \bar{r}}{dt^2} \right) dt = - \int_{t_1}^{t_2} m \frac{d \bar{\eta}(t)}{dt} \cdot \frac{d \bar{r}}{dt} dt$$

which can be substituted back into equation (3) to yield

$$\left. \frac{d I(\epsilon)}{d \epsilon} \right|_{\epsilon=0} \equiv \int_{t_1}^{t_2} \left\{ m \frac{d \bar{\eta}(t)}{dt} \cdot \frac{d \bar{r}}{dt} + \bar{F} \cdot \bar{\eta}(t) \right\} dt = 0 \quad (5)$$

Therefore, for the varied path, it follows from equation (5) that

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{t_1}^{t_2} \left\{ m \frac{d[\bar{r} + \epsilon \bar{\eta}(t)]}{dt} \cdot \frac{\partial}{\partial \epsilon} \left[ \frac{d[\bar{r} + \epsilon \bar{\eta}(t)]}{dt} \right] + \bar{F} \cdot \frac{\partial [\epsilon \bar{\eta}(t)]}{\partial \epsilon} \right\} dt \quad (6)$$

Integrating equation (6) with respect to  $\epsilon$  gives

$$I(\epsilon) = \int_{t_1}^{t_2} \left\{ \frac{m}{2} \left[ \frac{d[\bar{r} + \epsilon \bar{\eta}(t)]}{dt} \right]^2 + \bar{F} \cdot \epsilon \bar{\eta}(t) \right\} dt \quad (7)$$

Equation (7) is the desired form of Hamilton's Integral. The first term in the integrand of equation (7) is the kinetic energy of the particle along a varied path and the second term is the work done by the force system acting on the particle. Therefore, all that is required is the force  $\bar{F}$  be specified and the corresponding equations of motion for the pendulum dampers can be derived by use of equations (5)<sup>4</sup> through (7).

For a pendulum configuration of arbitrary geometry, such as in Figure 1, the following coordinate system can be established for a differential element of the entire pendulum; as shown in Figure 4.

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<sup>4</sup>Equation (5) is generally known as the first necessary condition that Hamilton's Integral be a minimum. See Ref. 4.

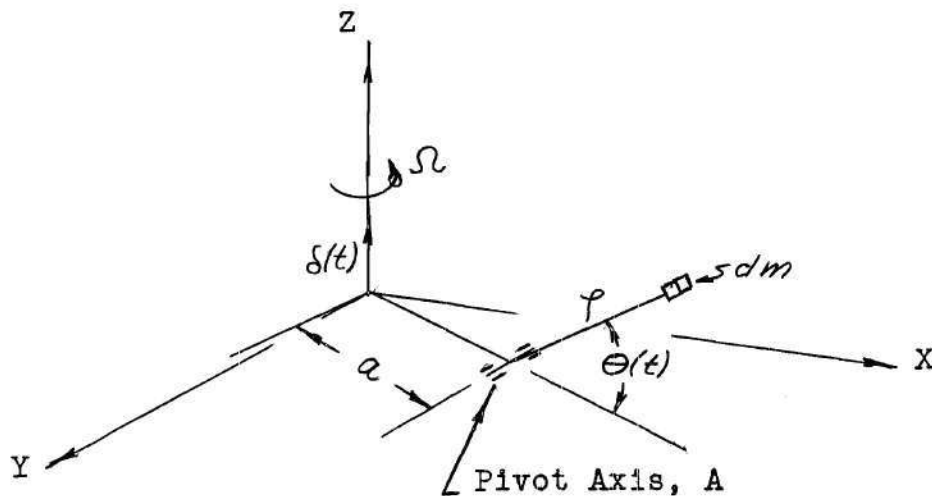


Figure 4. Coordinate System Used to Develop Equations of Motion for a Pendulum Damper Pivoted at a Fixed Radius "a" from a Rotating Shaft.

Let the vertical, or the longitudinal displacement of the rotor drive shaft be denoted by  $\delta(t)$ , and let the angular displacement of the pendulum be  $\Theta(t)$ . Define  $\dot{\Theta}_t = \frac{d\Theta(t)}{dt}$ ,  $\dot{\delta}_t = \frac{d\delta(t)}{dt}$ . Choose the xy-plane as shown in Figure 4 as the isopotential plane passing through the pivot axis A. Let  $\delta(t)$  and  $\Theta(t)$  be measured positive above the xy-plane.

Since the pendulum dampers must be positioned symmetrically around the rotor drive shaft. The equations of motion for any one pendulum will suffice. The acceleration, at any

time  $t$ , will be the same for each pendulum, and if there are  $n$  pendulums, each of mass  $m_1$ , then each pendulum would cancel  $1/n$  part of the forcing function. Hence, an equivalent non-dimensional force system, for force components in the axial direction, would be one in which there was but one pendulum, the mass of which was  $n$  times the mass of each symmetrically spaced pendulum; that is  $m = nm_1$ . The expression for the kinetic energy of the equivalent force system was derived as follows (refer to Figure 4).

The kinetic energy of a differential element of the pendulum is

$$dT = \frac{1}{2} \left\{ (\dot{\delta}_t)^2 + (r\dot{\theta}_t)^2 + 2r\dot{\theta}_t\dot{\delta}_t \cos \theta \right\} dm + \frac{1}{2} d(\xi - m) (\dot{\delta}_t)^2 + \frac{1}{2} (a + r \cos \theta)^2 \Omega^2 dm \quad (8)$$

Integrating over the volume of the pendulum, the total kinetic energy becomes

$$\begin{aligned} T = & \frac{1}{2} (\dot{\delta}_t)^2 \int_V dm + \frac{1}{2} (\dot{\theta}_t)^2 \int_V r^2 dm + \dot{\theta}_t \dot{\delta}_t \cos \theta \int_V r dm + \frac{1}{2} \xi (\dot{\delta}_t)^2 \\ & - \frac{1}{2} (\dot{\delta}_t)^2 \int_V dm + \frac{1}{2} (a\Omega)^2 \int_V dm + a\Omega^2 \cos \theta \int_V r dm \\ & + \frac{1}{2} \Omega^2 \cos^2 \theta \int_V r^2 dm \end{aligned} \quad (9)$$

where

$\xi \equiv$  Mass of the entire helicopter (including pendulum mass)

$$m \equiv \int_V dm$$

$$R_A m \equiv \int_V r dm$$

$$I_A \equiv \int_V r^2 dm$$

Equation (9) becomes

$$\begin{aligned} T = & \frac{1}{2} I_A (\dot{\theta}_t)^2 + R_A m \dot{\theta}_t \dot{\delta}_t \cos(\theta) + \frac{1}{2} \xi (\dot{\delta}_t)^2 + \frac{1}{2} (a \Omega)^2 m \\ & + a \Omega^2 R_A m \cos(\theta) + \frac{1}{2} I_A \Omega^2 \cos^2(\theta) \end{aligned} \quad (10)$$

If the work done by the force system on the equivalent pendulum is defined as

$$W(\epsilon_i, \epsilon_j) \equiv \bar{F}_i \eta_j(t) = N(t) \epsilon_2 \eta_2(t) - D_t(\theta_t) \epsilon_1 \eta_1(t) \quad (11)$$

$(i = 1, 2, 3 \dots n)$

where

$N(t)$  - The expression for the rotor thrust variation.

$D_t(\theta_t)$  - The torque arising from a dissipation function

$D(\theta_t)$  which acts at the pivot of the equivalent pendulum, as shown in Figure 2.

The change in gravitational potential energy for this system is negligibly small.

Substituting equation (10) and (11) into Hamilton's Integral of minimum energy, equation (7), gives

$$I(\epsilon, \epsilon_2) = \int_{t_1}^{t_2} \left\{ T(\epsilon, \epsilon_2) + W(\epsilon, \epsilon_2) \right\} dt \quad (12)$$

$$= \int_{t_1}^{t_2} \left\{ \frac{1}{2} I_A [\dot{\theta}_t + \epsilon_t \eta_t]^2 + R_A m [\dot{\theta}_t + \epsilon_t \eta_t] [\dot{\delta}_t + \epsilon_2 \eta_2] \cos(\theta + \epsilon, \eta) \right.$$

$$+ \frac{1}{2} \bar{\phi} [\dot{\delta}_t + \epsilon_2 \eta_2]^2 + \frac{1}{2} [a \Omega]^2 m + a \Omega^2 R_A m \cos(\theta + \epsilon, \eta) \left. \right\}$$

$$+ \frac{1}{2} I_A \Omega^2 \cos^2(\theta + \epsilon, \eta) + V(t) \epsilon_2 \eta_2 - D_t(\theta_t) \epsilon, \eta, \left. \right\} dt$$

Applying the first necessary condition that equation (12) be a minimum, as given in equation (5), it follows that

$$\frac{dI(0,0)}{d\epsilon_1} = \int_{t_1}^{t_2} \left\{ I_A \dot{\theta}_t \eta_t + R_A m [\eta_t \dot{\delta}_t \cos(\theta) - \eta_t \dot{\theta}_t \sin(\theta)] \right. \quad (13)$$

$$- a \Omega^2 R_A m \eta_t \sin(\theta) - I_A \Omega^2 \eta_t \cos(\theta) \sin(\theta)$$

$$\left. - D_t(\theta_t) \eta_t \right\} dt = 0$$



and

$$\frac{dI(0,0)}{d\epsilon_2} = \int_{t_1}^{t_2} \left\{ R_A m \theta_t \eta_{2t} \cos(\theta) + \xi \delta_t \eta_{2t} + N(t) \eta_2 \right\} dt \quad (14)$$

Integrating equations (13) and (14) by parts:

$$\frac{dI(0,0)}{d\epsilon_1} = \left[ I_A \theta_t + R_A m \delta_t \cos(\theta) \right] \eta_1(t) \Big|_{t_1}^{t_2} \quad (15)$$

$$+ \int_{t_1}^{t_2} \eta_1(t) \left\{ -I_A \theta_{tt} - R_A m \delta_{tt} \cos(\theta) + R_A m \theta_t \delta_t \sin(\theta) \right.$$

$$\left. - R_A m \theta_t \delta_t \sin(\theta) - a \Omega^2 R_A m \sin(\theta) \right.$$

$$\left. - I_A \Omega^2 \cos(\theta) \sin(\theta) - D_t(\theta_t) \right\} dt = 0$$

$$\frac{dI(0,0)}{d\epsilon_2} = \left[ R_A m \theta_t \cos(\theta) + \xi \delta_t \right] \eta_2(t) \Big|_{t_1}^{t_2} \quad (16)$$

$$+ \int_{t_1}^{t_2} \eta_2(t) \left\{ -R_A m \theta_{tt} \cos(\theta) + R_A m (\theta_t)^2 \sin(\theta) \right.$$

$$\left. - \xi \delta_{tt} + N(t) \right\} dt = 0$$

For the time  $t_1$  and the time  $t_2$ , the true path and the varied path coincide. Therefore  $\eta_1(t_1) = \eta_1(t_2) = \eta_2(t_1) = \eta_2(t_2) = 0$  and equations (15) and (16) reduce to

$$\int_{t_1}^{t_2} \eta_1(t) \left\{ -I_A \theta_{tt} - R_A m \delta_{tt} \cos \theta - a \Omega^2 R_A m \sin(\theta) - I_A \Omega^2 \cos(\theta) \sin(\theta) - D_t(\theta_t) \right\} dt = 0 \quad (17)$$

$$\int_{t_1}^{t_2} \eta_2(t) \left\{ -R_A m \theta_{tt} \cos(\theta) + R_A m [\theta_t]^2 \sin(\theta) - \xi \delta_{tt} + N(t) \right\} dt = 0 \quad (18)$$

From the fundamental lemma of calculus of variations<sup>5</sup>, if  $\eta_1(t)$  and  $\eta_2(t)$  are not equal to zero at all times, equations (17) and (18) become

$$I_A \theta_{tt} + R_A m \delta_{tt} \cos \theta + a \Omega^2 R_A m \sin(\theta) + I_A \Omega^2 \cos(\theta) \sin(\theta) + D_t(\theta_t) = 0 \quad (19)$$

$$R_A m \theta_{tt} \cos(\theta) - R_A m [\theta_t]^2 \sin(\theta) + \xi \delta_{tt} - N(t) = 0 \quad (20)$$

Equations (19) and (20) are the differential equations of motion for the nonconservative force system considered.

Tuning the Pendulum Damper. Up to this point the forcing function,  $N(t)$ , and the dissipation torque in the pivot axes,  $D_t(\theta_t)$ , are perfectly general. If the higher and sub-harmonic

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<sup>5</sup>See Reference 4, page 185.

variations in the helicopter blade thrust are neglected, the periodic variation in the rotor thrust can be expressed as

$$N(t) = N_b \cos(b\Omega t) \quad (21)$$

where b- Number of blades in the rotor

$\Omega$  - Average angular velocity of the rotor

$N_b$  - Magnitude of change in the rotor thrust having frequency  $b\Omega$ .

Let the dissipative torque be taken as viscous torque, or

$$D_t(\theta_t) \equiv \eta C, \theta_t \quad (22)$$

Equations (19) and (20) can then be written as follows:

$$I_A \theta_{tt} + R_A m \delta_{tt} \cos \theta + Q \Omega^2 R_A m \sin \theta + I_A \Omega^2 \cos(\theta) \sin(\theta) + \eta C, \theta_t = 0 \quad (23)$$

and equation (20) becomes

$$\xi \delta_{tt} + R_A m \theta_{tt} \cos \theta - R_A m (\theta_t)^2 \sin \theta - N_b \cos(b\Omega t) = 0 \quad (24)$$

By solving equation (24) for  $\delta_{tt}$  and substituting into equation (23), a second order differential equation in  $\theta$  is obtained, that is

$$\begin{aligned}
& I_A \theta_{tt} + \frac{(R_A m)^2}{\xi} (\theta_t)^2 \sin(\theta) \cos \theta - \frac{(R_A m)^2}{\xi} \theta_{tt} \cos^2 \theta \\
& - \frac{R_A m N_b}{\xi} \cos(b\Omega t) \cos \theta + R_A m a \Omega^2 \sin(\theta) \\
& + I_A \Omega^2 \cos(\theta) \sin(\theta) + n C_1 \theta_t = 0
\end{aligned} \quad (25)$$

Writing equation (24) in the following form

$$\xi \delta_{tt} + R_A m \theta_{tt} \cos(\theta) - R_A m (\theta_t)^2 \sin(\theta) = N_b \cos(b\Omega t) \quad (26)$$

and integrating twice with respect to time, the vertical excitation  $\delta(t)$  can be obtained. Integrating both sides of equation (26):

$$\int \frac{d}{dt} \left\{ \xi \delta_t + R_A m \theta_t \cos(\theta) \right\} dt = N_b \int \cos(b\Omega t) dt \quad (27)$$

Equation (27) reduces to

$$\xi \frac{d\delta}{dt} + R_A m \cos \theta \frac{d\theta}{dt} = \frac{N_b}{b\Omega} \sin(b\Omega t) + K_1 \quad (28)$$

Likewise, integrating both sides of equation (28) with respect to time:

$$\frac{E}{\ell} \int \frac{d\delta}{dt} dt + R_A m \int \cos(\theta) \frac{d\theta}{dt} dt = + \frac{N_b}{b\Omega} \int \sin(b\Omega t) dt + K_1 \int dt \quad (29)$$

Equation (29) then becomes

$$\frac{E}{\ell} \delta(t) + R_A m \sin \theta(t) = - \frac{N_b}{b^2 \Omega^2} \cos(b\Omega t) + K_1 t + K_2 \quad (30)$$

It has already been specified that the helicopter is in unaccelerated flight. The displacement  $\delta(t)$  is known to vary periodically about the isopotential plane, as shown in Figure 4. Therefore, as time increases, the mean value of the total energy remains constant. This implies that the constant of integration  $K_1$ , in equation (30), must be zero for all time. If for time equal to zero, the  $\delta(t)$  motion is zero, then the initial angle of the pendulum damper can be determined. That is

$$K_2 = -R_A m \sin(\theta_0) \quad (31)$$

where  $\theta(0) = \theta_0$  for  $t = 0$

Finally, the  $\delta(t)$  motion can be expressed as

$$\delta(t) = \frac{-N_b}{b^2 \Omega^2 \ell} \cos(b\Omega t) - \frac{R_A m}{\ell} [\sin \theta(t) - \sin \theta_0] \quad (32)$$

Upon examination of equation (32) it would appear that there should be some form of  $\Theta(t)$ , for a given set of parameters, that would make the amplitude  $\delta(t)$  a minimum. It can be shown (1) that for small amplitude displacements of a properly designed pendulum damper, the vertical displacement  $\delta(t)$  can be reduced to zero for all times. Unfortunately, the corresponding mass of the small amplitude pendulum damper is prohibitively large for helicopter use.

In general, owing to the non-linearity of  $\Theta(t)$  for large amplitude motion of the pendulums,  $\delta(t)$  cannot be reduced to zero for all times. However, the general non-linear waveform of  $\Theta(t)$  that will minimize  $\delta(t)$  can be determined as follows. Upon further examination of equation (32), it appears that the  $\sin \Theta(t)$  term must correspond to a frequency of  $b\Omega$  when the pendulum damper is turned to remove all of the axial vibration  $\delta(t)$ . If the amplitude of  $\Theta(t)$  is confined to the range  $-\pi/2 \leq \Theta(t) \leq \pi/2$ , the question then arises: does  $\sin \Theta(t) = \sin(b\Omega t)$  for all  $t$ , when  $\Theta(t)$  is a triangular waveform, as shown in Figure 5. If so, then the  $\sin \Theta(t)$  term would behave exactly as  $\sin(b\Omega t)$  and with the proper phase relationship  $\delta(t)$  could be made zero for all time, provided  $-\pi/2 \leq \Theta(t) \leq \pi/2$ . This can be shown as follows (see next page).

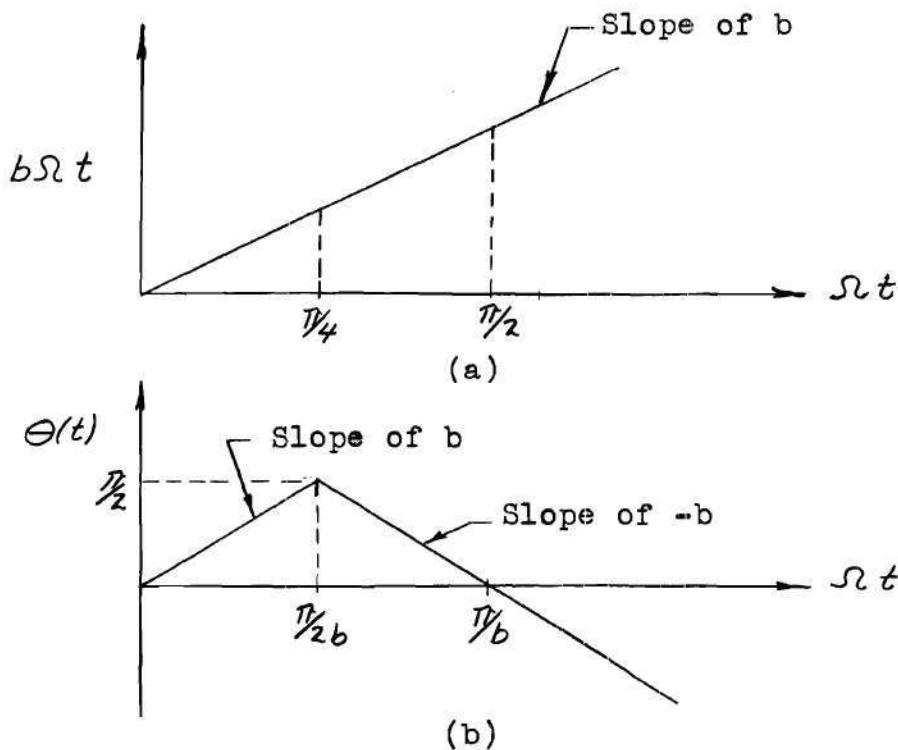


Figure 5. (a) Linear Variation of  $b\Omega t$  with Time; (b) Triangular Variation of  $\Theta(t)$  with  $\Omega t$ .

The equation of the curve in Figure 5-b between  $\frac{\pi}{2b} \leq \Omega t \leq \frac{\pi}{b}$  is  $\Theta(t) = \pi - b\Omega t$ .

Hence  $\sin \Theta(t) = \sin(\pi - b\Omega t) = \sin(\pi) \cos(b\Omega t) - \cos(\pi) \sin(b\Omega t)$

and therefore  $\sin \Theta(t) = \sin(b\Omega t)$  for  $b = 1, 2, 3 \dots n$ .

By representing  $\Theta(t)$  as a triangular waveform,  $\delta(t)$ , in equation (32), is reduced to zero for all time. A similar argument can be given for  $\Omega t > \pi/b$ .

However, a triangular waveform would not yield a physically possible solution to equation (25). Therefore, for a minimum amplitude of  $\delta(t)$ , the waveform of  $\Theta(t)$  would have to be a smooth curve which closely approximates a triangular waveform, as shown in Figure 6. That is,

$$\sin \Theta(t) \approx \sin(b\Omega t) = \cos(b\Omega t - \pi/2)$$

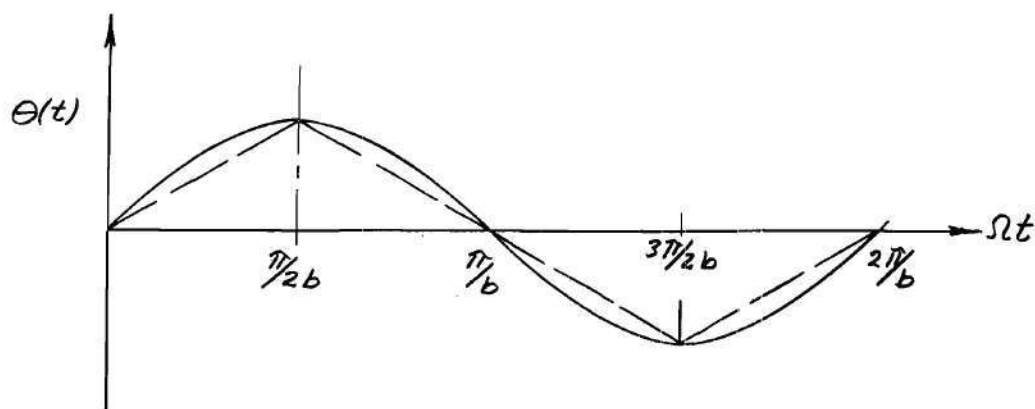


Figure 6. Optimum Waveform of Pendulum Dampers When Tuned to Reduce Amplitude of  $\delta(t)$  Motion to a Minimum.



Equations (25) and (32) can be nondimensionalized in the following manner, and corresponding solutions for  $\delta(t)$  and  $\Theta(t)$  obtained from a computer. Multiplying equation (25) through by  $1/R_A^2 m$ , which is never zero, gives

$$\begin{aligned} \left[ \frac{I_A}{R_A^2 m} \right] \Theta_{tt} + \frac{m}{\xi} \left\{ [\Theta_t]^2 \sin(\Theta) \cos(\Theta) - \Theta_{tt} \cos^2(\Theta) \right\} \\ + \left[ \frac{N_b}{R_A \xi \Omega^2} \right] \Omega^2 \cos(b\Omega t) \cos(\Theta) + \left[ \frac{a}{R_A} \right] \Omega^2 \sin(\Theta) \\ + \left[ \frac{C, n}{R_A^2 m \Omega} \right] \Omega \Theta_t + \Omega^2 \sin(\Theta) \cos(\Theta) = 0 \end{aligned} \quad (33)$$

Then, defining the nondimensional time,  $t'$ , as

$$t' \equiv \Omega t \quad (34)$$

equation (33) can be written in the following nondimensional form:

$$\begin{aligned} \left[ \frac{I_A}{R_A^2 m} \right] \Omega^2 \Theta'_{tt} + \left[ \frac{m}{\xi} \right] \Omega^2 \left\{ [\Theta'_t]^2 \sin(\Theta') \cos(\Theta') - \Theta'_{tt} \cos^2(\Theta') \right\} \\ + \left[ \frac{N_b}{R_A \xi \Omega^2} \right] \Omega^2 \cos(bt') \cos(\Theta') + \left[ \frac{a}{R_A} \right] \Omega^2 \sin(\Theta') \\ + \left[ \frac{C, n}{R_A^2 m \Omega} \right] \Omega^2 \Theta'_t + \Omega^2 \sin(\Theta') \cos(\Theta') = 0 \end{aligned} \quad (35)$$

where the primes denote the nondimensional value of the function. If a nondimensional length,  $\delta'$ , is defined as

$$\delta' = \delta / R_A$$

then equation (32) can be written as

$$\delta' = \frac{1}{b^2} \left[ \frac{-N_b}{\xi \Omega^2 R_A} \right] \cos(2t') - \left[ \frac{m}{\xi} \right] \left[ \sin(\theta') - \sin \theta_0' \right] \quad (36)$$

Define the following nondimensional parameters as:

$$A \equiv \left[ \frac{I_A}{R_A^2 m} \right]$$

$$C \equiv \left[ \frac{-N_b}{R_A \xi \Omega^2} \right]$$

$$E \equiv \left[ \frac{C, n}{R_A^2 m \Omega} \right]$$

$$B \equiv \left[ \frac{m}{\xi} \right]$$

$$D \equiv \left[ \frac{a}{R_A} \right]$$

$$m \equiv n m,$$

Equations (35) and (36) then reduce to, respectively,

$$A \theta_{tt}' + B \left\{ [\theta_t']^2 \sin \theta' \cos(\theta') - \theta_{tt}' \cos^2 \theta' \right\} - C \cos(2t') \cos(\theta') \quad (37)$$

$$+ D \sin \theta' + E \theta_t' + \cos(\theta') \sin(\theta') = 0$$

$$\delta' = \frac{C}{b^2} \cos(2t') - B \left\{ \sin(\theta') - \sin \theta_0' \right\} \quad (38)$$

Equations (37) and (38) are readily applicable to programming for a computer.

### CHAPTER III

#### SOLUTION BY ANALOG COMPUTER

An analog computer was chosen to solve the non-linear differential equations of motion for the pendulum rather than a digital computer because of the relative ease with which results from the analog computer could be observed and altered.

A Berkely EASE analog computer was used to solve equation (37), the non-linear differential equation that described the motion of the tuned pendulum damper configuration. The corresponding solution to the equation was used to determine the longitudinal displacement of the rotating shaft, and peak amplitudes were observed and recorded from an Epsco digital volt meter, Model DV-803. A Du Mont, 350-R Cathode Ray Oscilloscope was used to keep a running check on the output of the amplifiers during computation. A six channel Offner Electronics Dynagraph Recorder was used to make permanent recordings of the periodic displacements and velocity curves for the pendulum and the drive shaft motions.

Equations (37) and (38) were adapted to the computer in the following manner. For scaling purposes, an average

rotor angular velocity of 300 rpm, or  $\Omega = 10\pi$  rad./sec was used. It should be noted that the specified value,  $\Omega = 10\pi$ , was simply for scale factoring. This does not imply that  $\Omega = 10\pi$  is the only value of  $\Omega$  for which the subsequent solutions are valid. For the computer program that was used, it was optimum to let  $\tau = 2t$ , where  $\tau$  is computer time and  $t$  is real time. Since  $t' = 10\pi t$ , from equation (34), and  $\tau = 2t$ , then  $t' = 5\pi\tau$ . Consider a rotor having two blades, that is,  $b = 2$ ; then, equation (37) can be written in terms of  $\tau$ . That is for

$$\theta'_t = \frac{d\theta}{dt'} = \frac{d\theta}{d\tau} \cdot \frac{d\tau}{dt'} = \frac{1}{5\pi} \cdot \frac{d\theta}{d\tau} = \frac{1}{5\pi} \theta_\tau$$

$$\theta'_{tt} = \frac{d^2\theta}{dt'^2} = \frac{d^2\theta}{d\tau^2} \cdot \frac{d^2\tau}{dt'^2} = \frac{1}{25\pi^2} \frac{d^2\theta}{d\tau^2} = \frac{1}{25\pi^2} \theta_{\tau\tau}$$

Equation (37) can now be written as

$$-\frac{A}{10} \left\{ 1 - \frac{B}{A} \cos^2 \theta \right\} \theta_{\tau\tau} = \left\{ \frac{B}{10} [\theta_\tau]^2 + 2.5\pi^2 \right\} \sin(\theta) \cos(\theta) \quad (39)$$

$$- [2.5\pi^2] \left\{ C \cos(10\pi\tau) \right\} \cos \theta + [2.5\pi^2] D \sin(\theta) + \left[ \frac{\pi}{2} \right] E \theta_\tau$$

which is the non-dimensional form of the equation of motion as solved by the analog computer used in this study.

Consider the following range of parameters for equation (39)

$$1 < A \leq 2.000$$

$$0 < B \leq 0.005$$

$$0 \leq C \leq 0.010$$

$$0 < D \leq 4.000$$

$$0 \leq E \leq 0.500$$

For the above range of parameters it can be seen that the term  $\frac{B}{A} \cos^2(\theta) \ll 1$ . Hence, neglecting the term  $\frac{B}{A} \cos^2(\theta)$  equation (39) can be written as

$$\begin{aligned} -\frac{A}{10} \theta_{tt} = & \left\{ \frac{B}{10} [\theta_t]^2 + 2.5 \pi^2 \right\} \sin(\theta) \cos(\theta) \\ & - 2.5 \pi^2 C \cos(10 \pi t) \cos(\theta) \\ & + 2.5 \pi^2 D \sin(\theta) + \frac{\pi}{2} E \theta_t \end{aligned} \quad (40)$$

Correspondingly, equation (38) can be written as

$$50 S' = \frac{50}{4} C \cos(10 \pi t) - 50 B [\sin(\theta) - \sin(\theta_0)] \quad (41)$$

where  $b = 2$ .

Equations (40) and (41) were subsequently programmed and the electronic analog simulation of the mathematical

equations is shown in Figure 7. In general, the procedure used to obtain the minimum  $\delta(t)$  was to observe the Lissajous pattern between  $\Theta(t)$  and the forcing function,  $\cos(10\pi t)$ , on the oscilloscope and to vary the different parameters, A, B, C, D, and E, until the Lissajous pattern stabilized as a straight line at forty-five degrees to the horizontal axis of the oscilloscope. Then, when the modulus, or the length of the line, remained constant, the pendulum was tuned corresponding to the parameters that stabilized the Lissajous pattern.

It is important to note that not just any combination of the order of parameter adjustments will stabilize the Lissajous pattern. After numerous attempts the following sequence proved to be a satisfactory method for tuning the pendulum damper. An upper value for the parameter B was selected; then D was set at zero and E was adjusted to obtain a steady waveform for  $\Theta(t)$ . Next, the parameter C was increased to a desired value by increments, adjusting E each time in order to maintain a steady state  $\Theta(t)$ . The parameter A was then adjusted until the Lissajous underwent a minimum precession and elongation. Finally D was adjusted to steady the pattern, and B was then used to narrow the ellipse to a line. The Lissajous pattern that corresponded to a trace that appeared as a straight line at a slope of forty-five degrees was the final indication that the pendulum



was tuned. The ordinate measure at the tip of the Lissajous line trace was the peak value of  $\Theta(t)$ .

The combination of parameters that tuned the pendulum damper for different peak amplitudes of oscillation can be found in Table 1.

Table 1. Parameters that Correspond to Solutions of  $\Theta(t)$  that Minimize  $\delta(t)$

Maximum Amplitude of $\Theta(t)$	Parameter A	Parameter B	Parameter C	Parameter D	Parameter E
$\pm 30^\circ$	1.114	0.0030	0.00885	3.6000	0.3606
$\pm 60^\circ$	1.092	0.0021	0.00400	4.0513	0.3558

Where again

$$A = \frac{I_A}{R_A^2 m}$$

$$C = \frac{-N_2}{R_A \xi \Omega^2}$$

$$E = \frac{n C_1}{R_A^2 m \Omega}$$

$$B = \frac{m}{\xi}$$

$$D = \frac{a}{R_A}$$

$$m = nm,$$

The values for the parameters in Table 1 correspond to a reduction in  $\delta(t)$  by the amounts shown in Table 2.



Table 2. Effectiveness of Pendulum Dampers

Maximum Amplitude of $\Theta(t)$	Maximum Amplitude of $\delta(t)$ without Pendulum Dampers	Maximum Amplitude of $\delta(t)$ with Pendulum Dampers	Percent Reduction $\delta(t)$
$\pm 30^\circ$	0.00221	$\sim 0.00079$	$\sim 65.00$
$\pm 60^\circ$	0.00100	$\sim 0.00000$	$\sim 100.00$

One reason for so few sets of solutions is that in order to tune the damper, or, in analogy, to stabilize the Lissajous pattern, it was found that the parameter settings for A, B, C, D, and E, were quite critical. In other words, for large amplitude oscillations the tuning was sharp even with the presence of damping. This is somewhat contrary to the theory of small oscillations when damping is present. It is believed that the sensitivity of the damping, E, required to obtain a steady state solution for  $\Theta(t)$  to minimize the  $\delta(t)$  motion can be attributed to the fact that the rotational speed of the rotor drive shaft was held constant. Hence, the total energy was not conserved for the system and stray harmonics were induced into the pendulum oscillatory motion. Therefore, a precise amount of dissipation was probably required to minimize the

stray harmonic motion and allow the damper to be tuned to the forcing function of prescribed frequency.

The accuracy of the parameters obtained from the analog computer is believed to be well within engineering tolerances.

An analog solution for small amplitude displacements was found to be in good agreement with the small amplitude solution obtained in closed form in Reference 1. The basic results obtained from Reference 1 are listed below in Table 3, and the corresponding geometry shown in Figure 8.

Table 3. Results Obtained from Reference 1 for  
Small Amplitude Displacements of a  
Tuned Pendulum Damper

Max. Ampli- tude for $\Theta(t)$	Parameter A	Parameter B	Parameter C	Parameter D	Parameter E	Percent Reduc- tion in $\delta(t)$
$\pm 3.16$	1.010	0.0343	0.0079	3.04	0.00	100.00

It is important to note the weight reduction in the pendulum dampers between the large and the small amplitude solutions of  $\Theta(t)$  that correspond to a one hundred percent reduction in  $\delta(t)$ . The weight for the pendulum damper corresponding

to a maximum amplitude of  $\pm 63^\circ$  is approximately one-tenth the weight of the pendulum damper with a maximum amplitude of  $\pm 3^\circ.16$  for a point mass having the same pivot radius.

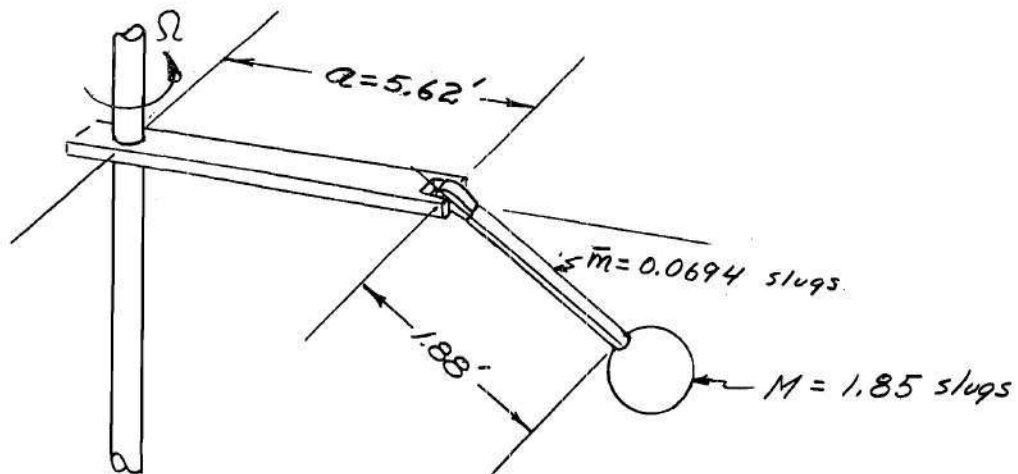


Figure 8. Sketch of Geometrical Configuration of Pendulum Damper Used in Reference 1.

## CHAPTER IV

THE DESIGN OF A TUNED PENDULUM DAMPER  
SYSTEM OF  $n$  PENDULUMS

The problem is to design one of  $n$  tuned pendulum dampers that will remove  $1/n^{\text{th}}$  of the exciting force  $N_2$ . From Table 1, for  $\Theta(t)$  with a peak amplitude of  $\pm 1.10$  radians the corresponding non-dimensional combination of parameters was obtained.

$$A = 1.092$$

$$D = 4.0513$$

$$B = 0.0021$$

$$E = 0.3558$$

$$C = 0.004$$

One procedure that can be used to design a pendulum damper system with  $n$  pendulum dampers is as follows:

Given:  $\Omega = 30$  radians/sec

$$\xi = 5000/g \text{ slugs}$$

$$N_2 = -600 \#$$

a.) Compute the mass of a pendulum damper.

$$B = \frac{m}{\xi} = 0.0021$$

For one pendulum then

$$\frac{B}{n} = \frac{0.0021}{n} = \frac{m_1}{\xi}$$

or 
$$m_1 = \frac{\xi}{n} (.0021) = \frac{(5000)(.0021)}{gn} = \frac{10.5}{gn}$$

Let  $g = 32.2 \text{ ft/sec}^2$  and  $n = 2$ , where again  $n$  is the number of pendulums. Hence,

$$m_1 = \frac{(10.5)}{(2)(32.2)} = 0.163 \text{ slugs}$$

b.) Compute the distance from the pivot axes to the center of gravity of the pendulums.

$$c = \frac{N_2}{R_A \cancel{f} \Omega^2} = 0.004$$

or

$$R_A = \frac{N_2}{(0.004) \cancel{f} \Omega^2} = \frac{(600)(32.2)}{(0.004)(5000)(30)^2} = 1.07 \text{ ft}$$

c.) Compute the distance from the drive shaft axis to the pivot axis.

$$D = \frac{a}{R_A} = 4.05$$

or

$$a = (4.05)(1.07) = 4.35 \text{ ft}$$

d.) Compute the viscous torque required in the pivot axes.

$$E = \frac{C_1}{R_A^2 m_1} = 0.356$$

or

$$C_1 = (0.356) R_A^2 m_1$$

$$= (0.356)(1.07)^2(0.163) = 2.0 \text{ ft# sec.}$$

Hence,

$$D_t(\theta_t) = (2.0)\theta_t \text{ ft}\#$$

e.) Compute the mass moment of inertia of each pendulum damper about the pivot axis

$$A = \frac{I_A}{R_A^2 m_1} = (1.092)$$

$$I_A + (1.092) R_A^2 m_1 = (1.092)(1.07)^2(0.163) = 0.205 \text{ slugs ft}^2$$

It is estimated that as much as five percent error is possible in each value of the parameters obtained from the analog computer. The results obtained in this example problem should not be taken as precise quantitative values.

## CHAPTER V

### CONCLUSIONS

Periodic excitations imparted axially to a rotor shaft of a helicopter apparently can be removed by attaching light-weight tuned pendulum dampers to the rotating shaft. Like small amplitude tuning without damping, the large amplitude tuning with damping was found to be quite sharp.

The viscous torque required to obtain the solutions for the tuned pendulum dampers is relatively small and it is believed that "fin" type air cooling should be adequate for most viscous type damping located in the pivot axes of the pendulums.

A distinct advantage is obtained in the generality of the solutions for the tuned pendulum dampers in that a wide variety of geometrical configurations can be designed from the corresponding combination of parameters. It is estimated that as much as five percent error is possible in each value of the parameters obtained from the analog computer.

## CHAPTER VI

## RECOMMENDATIONS

No experimental verification of the analog results from this investigation has yet been made and it appears that such verification would be worth while. A better approximation to the parameter combinations of the second harmonic tuned pendulum damper solutions could be made on a digital computer. In addition, a more practical analytical investigation would be to design a pendulum damper configuration having a moving instantaneous center of oscillation rather than a fixed pivot axis. This system could then be tuned for variable amplitude oscillatory motion of the pendulum dampers.

It might also be worth while to consider the helicopter fuselage as a free-free, non-uniform beam instead of a lumped mass. It is believed that an analysis of only the fundamental mode of vibration would be sufficient to yield accurate results.



## A P P E N D I X

# LINEAR FORM OF THE EQUATIONS OF MOTION FOR SECOND HARMONIC TUNED PENDULUM DAMPERS

By making the assumption that  $\sin \Theta(t) \approx \Theta(t)$  and  $\cos \Theta(t) \approx 1.0$ , and neglecting higher ordered terms of  $\Theta$ , the resulting linear form of equations (23) and (24) became respectively,

$$I_A \Theta_{tt} + R_A m \delta_{tt} + a \Omega^2 R_A m \Theta + I_A \Omega^2 \Theta + n C_1 \Theta_t = 0 \quad (42)$$

$$\xi \delta_{tt} + R_A m \Theta_{tt} - N_b \cos(b \Omega t) = 0 \quad (43)$$

Equations (42) and (43) correspond exactly to equations (6-A) and (5-A) respectively, found on page 7 of reference 1. The values of the parameters  $R_A$ ,  $m$ ,  $I_A$ ,  $a$ ,  $N_b$ ,  $C_1$  and  $\xi$  found in reference 1 are as follows (refer to Figure 8).

$$R_A = \frac{B(M + \bar{m}l/2)}{(M + )}$$

$$I_A = \frac{1}{3} \bar{m} l^3 + M b^2$$

$$N_b = -N_0$$

$$C_1 = 0$$

$$m = M + \bar{m} \ell$$

$$\bar{\xi} = \xi$$

$$a = a$$

$$b = P$$

Equations (43) and (42) then reduce to, respectively,

$$\xi \delta_{tt} + \ell [M + \bar{m} \ell_2] \theta_{tt} = -N_0 \cos(Pt + \psi_0) \quad (44)$$

$$[M + \bar{m} \ell_3] \ell^2 \theta_{tt} + [M + \bar{m} \ell_2] \ell \delta_{tt} \quad (45)$$

$$= \{ [M + \bar{m} \ell_2] a + [M + \bar{m} \ell_3] \ell \} \ell \Omega^2$$

Assuming

$$\delta = \delta_0 \cos(Pt - \psi_0)$$

$$\theta = \theta_0 \cos(Pt - \psi_0)$$

and substituting into equations (44) and (45) gives

$$\delta_0 = \frac{\{ [M + \bar{m} \ell_3] \ell P^2 - [M + \bar{m} \ell_2] a \Omega^2 - [M + \bar{m} \ell_3] \ell \Omega^2 \} N_0}{\xi P^2 \{ [M + \bar{m} \ell_2] a + [M + \bar{m} \ell_3] \ell \} \Omega^2 - \xi \{ P^4 \ell [M + \bar{m} \ell_3] + [M + \bar{m} \ell_2]^2 \ell P^8 \}} \quad (46)$$

$$\alpha_0 = \frac{-[M + \bar{m}l_2]N_0}{\xi \{S^2[M + \bar{m}l_2]a + [M + \bar{m}l_2]l\Omega^2 - [M + \bar{m}l_2]lP^2\} + [M + \bar{m}l_2]^2 l P^2} \quad (47)$$

If the rotor shaft does not vibrate,  $\delta_0$  must equal zero, therefore, from equation (46)

$$[M + \bar{m}l_3]lP^2 - \{[M + \bar{m}l_2]a + [M + \bar{m}l_2]l\}\Omega^2 = 0 \quad (48)$$

The tuning equation can then be obtained from equation (48); that is, solving for 'P'

$$P = \Omega^2 \left\{ \frac{[M + \bar{m}l_2]a + [M + \bar{m}l_2]l}{[M + \bar{m}l_3]l} \right\} \quad (49)$$

The amplitude equation becomes

$$\theta_0 = \frac{-N_0}{[M + \bar{m}l_2]lP^2} \quad (50)$$

Equation (49) is the same as equation (11-A), and equation (50) is the same as equation (12-A) from reference 1. In general, by specifying  $P, \Omega, \ell$ , and "a", one can solve equation (49) for  $M$  in terms of  $\bar{m}$ . Equation (50) can be solved in a similar manner by specifying  $N_0$ .

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